Tutorial 6:

Preliminary:

Outstanding Balance:

Set $OB_0 = L = K_1 v + K_2 v^2 + \cdots + K_n v^n$

after first payment K_1, $OB_1 = OB_0(1+i) - K_1$,

after second payment K_2, $OB_2 = OB_1(1+i) - K_2 = (OB_0(1+i) - K_1)(1+i) - K_2$

$= OB_0(1+i)^2 - K_1(1+i) - K_2$

$\vdots$

after t^{th} payment K_t, $OB_t = OB_{t-1}(1+i) - K_t$.

$= OB_0(1+i)^t - K_1(1+i)^{t-1} - \cdots K_{t-1}(1+i) - K_t$

if $K_1 = K_2 = \cdots = K_t = K$, (retrospective) $= OB_0(1+i)^t - K s_{\overline{t|}i} = K a_{\overline{n-t|}i}$ (prospective)

Interest paid: $I_t = OB_{t-1} \times i$

Principal repaid: $PR_t = OB_{t-1} - OB_t = OB_{t-1} - OB_{t-1}(1+i) + K_t = K_t - OB_{t-1} \cdot i = K_t - I_t$.

if $K_1 = K_2 = \cdots = K_n = K$, $I_t = K a_{\overline{n-t+1|}i} \times i = K(1 - v^{n-t+1})$

$PR_t = K - I_t = K v^{n-t+1} \Rightarrow \frac{PR_t}{PR_n} = \frac{K v^{n-t+1}}{K v^{n-n+1}} = v^{n-t} = (1+i)^{t-n}$

Exercises:

3.1.1.

$K_1 = K_2 = \cdots = K_5 = 1000$, $K_6 = K_7 = \cdots = K_{10} = 500$, $i = 10\%$ "start a new from now".

(i) $L = 1000\, a_{\overline{5|}10\%} + 500\, a_{\overline{5|}10\%} \cdot v^5 = 4{,}967.68$

(ii) $OB_3 = OB_0(1+i)^3 - 1000\, s_{\overline{3|}i} = 4967.68 \times (1.1)^3 - 1000\, s_{\overline{3|}10\%} = 3301.98$.

(iii) $I_4 = OB_3 \times i = 3301.98 \times 0.1 = 330.20$.

$PR_4 = K_4 - I_4 = 1000 - 330.20 = 669.80$.

(iv) $OB_8 = 500\, a_{\overline{2|}10\%} = 867.77$.

3.1.4.

(i) $L = 20{,}000$, $n = 4 \times 12 = 48$, $i^{(12)} = 6\%$. $i = 0.5\%$.

assume monthly payment K.

$L = K a_{\overline{12|}0} + K v_0^{12} a_{\overline{36|}0.5\%} = 44.87K \Rightarrow K = 445.72$.

$OB_{12} = K a_{\overline{36|}0.5\%} = 14{,}651$.

(ii) $j_1^{(12)} = 3\%$, $j_1 = 0.25\%$, $j_2^{(12)} = 5\%$, $j_2 = 0.42\%$, assume payment P.

$P a_{\overline{12|}0.25\%} + P v_{0.25\%}^{12} a_{\overline{36|}0.42\%} = L \Rightarrow P = 452.61$.

$OB_{12} = P a_{\overline{36|}0.42\%} = 15{,}102$.

3.2-5.

(a) $n=5\times2=10$, "immediate". $K=200$.

$PR_1 = 156.24 = 200v_j^{n-t+1} = 200v_j^{10} \Rightarrow v_j^{10} = 0.7812 \Rightarrow j = 0.025$, semiannual rate.

monthly rate $(1+i)^{12} = (1+j)^2 \Rightarrow i = (1+j)^{\frac{1}{6}} - 1 \Rightarrow i^{(12)} = 12\times i = 0.0495$.

(b) $n=48$, $K=200$, $I_1+I_2+\cdots+I_{12} = 983.16$. $PR_{37}+PR_{38}+\cdots+PR_{48} = 2215.86$.

$PR_1+PR_2+\cdots+PR_{12} = (K-I_1)+(K-I_2)+\cdots+(K-I_{12}) = 2400-983.16 = 1416.84$.

$$\frac{PR_{37}+\cdots+PR_{48}}{PR_1+\cdots+PR_{12}} = (1+i)^{36} = \frac{2215.86}{1416.84} \Rightarrow i = 0.0125 \Rightarrow i^{(12)} = 0.15.$$

Tutorial : 3.1.1, 3.1.4, 3.2.5 ; Problem Set: 3.1.2, 3.1.6, 3.1.9, 3.1.10, 3.2.1, 3.2.2, 3.2.3, 3.2.4

3.1.2.

$OB_t = OB_0(1+i)^t - K_1(1+i)^{t-1} - K_2(1+i)^{t-2} - \cdots - K_{t-1}(1+i) - K_t$

$= (K_1 v + K_2 v^2 + \cdots + K_n v^n)(1+i)^t - K_1(1+i)^{t-1} - \cdots - K_t$

$= K_{t+1} v + K_{t+2} v^2 + \cdots + K_n v^{n-t}$

monthly rate $i = \frac{9\%}{12} = 0.75\%$, $v_{0.0075} = \frac{1}{1+0.75\%}$

Final 20 payments $K_{41} = 1000(1-2\%)^{40}$, $K_{42} = 1000(1-2\%)^{41}$, $\ldots$, $K_{60} = 1000(1-2\%)^{59}$

$OB_{40} = 1000 \cdot 0.98^{40} \cdot v_{0.0075} + 1000 \cdot 0.98^{41} \cdot v^2_{0.0075} + \cdots + 1000 \cdot 0.98^{59} \cdot v_{0.0075}$

$= 1000 \cdot 0.98^{40} v_{0.0075} [1 + 0.98 v_{0.0075} + 0.98^2 v^2_{0.0075} + \cdots + 0.98^{19} v^{19}_{0.0075}]$

$= 1000 \cdot 0.98^{40} \cdot v_{0.0075} \cdot \frac{1-(0.98 v_{0.0075})^{20}}{1-0.98 v_{0.0075}} = 6889.$

3.1.6.

$L = OB_0 = (OB_0 - OB_1) + (OB_1 - OB_2) + \cdots + (OB_{n-1} - OB_n)$

$= PR_1 + PR_2 + \cdots + PR_n$

$= (K_1 - I_1) + (K_2 - I_2) + \cdots + (K_n - I_n) = K_T - I_T$

$OB_0 = L.$

outstanding balance $OB_{t+1} = OB_t(1+i) - K_{t+1}$

principal repaid $PR_{t+1} = K_{t+1} - I_{t+1}$

$I_{t+1} = OB_t \times i.$

$PR_{t+1} = OB_t - OB_{t+1}$

3.1.9.

After first ten years, we still should repay $1000 = OB_0$.

payments $1.5 I_1, 1.5 I_2, \ldots, 1.5 I_{10}, \underbrace{X, X, \ldots, X}_{10}$

$OB_{10} = L$, $OB_{11} = OB_{10}(1+i) - 1.5 I_1 = OB_{10}(1+i) - 1.5 OB_{10} \cdot i = OB_{10}(1-0.5i) = L(1-0.5i).$

$OB_{12} = OB_{11}(1+i) - 1.5 I_2 = OB_{11}(1+i) - 1.5 OB_{11} \cdot i = OB_{11}(1-0.5i) = L(1-0.5i)^2.$

$OB_{20} = L(1-0.5i)^{10}$

After the next ten years, we need to pay $OB_{20} = L(1-0.5i)^{10} = 1000(1-0.05)^{10} = 598.74$

$X\, a_{\overline{10}|0.1} = 598.74, \Rightarrow X = 97.44.$

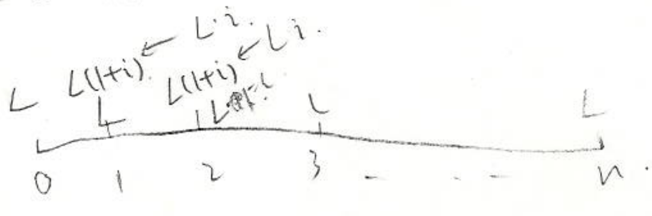
3.1.10.

If everytime we only pay for interest, then the all of the payments are $L \cdot i$.

$L \cdot i \cdot a_{\overline{n}|i} + L \cdot v^n = L \cdot i \cdot \frac{1-v^n}{i} + L v^n = L.$

$L = L \cdot i \cdot a_{\overline{n-1}|i} + \cancel{L(1+i) v^n}$

$L(1+i) = L + L \cdot i$.



$L \cdot v^n$

$+ L i \cdot a_{\overline{n}|i}$

3.2.1.

$L = K \cdot a_{\overline{n}|i}$.

$L(1+i)^t = K \cdot a_{\overline{n}|i}(1+i)^t = K \cdot \frac{1-v^n}{i} \cdot (1+i)^t = K \cdot \frac{(1+i)^t - v^{n-t}}{i} = K\left[\frac{(1+i)^t-1}{i} + \frac{1-v^{n-t}}{i}\right]$

$= K[s_{\overline{t}|i} + a_{\overline{n-t}|i}] = Ks_{\overline{t}|i} + Ka_{\overline{n-t}|i}$ $(\therefore L(1+i)^t - Ks_{\overline{t}|i} = Ka_{\overline{n-t}|i}.)$

By definition, $OB_t = L(1+i)^t - Ks_{\overline{t}|i} = Ka_{\overline{n-t}|i}$.

$L(1+i)^t - Ks_{\overline{t}|i} = L(s_{\overline{t}|i} \cdot i + 1) - Ks_{\overline{t}|i} = (Li-K)s_{\overline{t}|i} + L = L + (I_1 - K)s_{\overline{t}|i}$

$= L - PR_1 s_{\overline{t}|i}$

$\left(\frac{(1+i)^t-1}{i} = s_{\overline{t}|i} \Rightarrow (1+i)^t = s_{\overline{t}|i} \cdot i + 1\right)$

3.2.2.

Quarterly payment is $\frac{3000}{a_{\overline{12}|.02}} = 283.68$. Total interest paid $283.68 \times 12 - 3000 = 404.15$.

The original method has larger payments in the beginning, which reduces more outstanding balance so that the interest is also decreasing. So they don't need to pay that much interest.

3-2-3.

$OB_0 - OB_1 = PR_1 \Rightarrow OB_0 = PR_1 + OB_1 = 706 + 156 = 862$.

$I_1 = OB_0 \cdot i \Rightarrow i = \frac{I_1}{OB_0} = \frac{43.10}{862} = 0.05$, $I_2 = OB_1 \cdot i = 706 \times 0.05 = 35.30$.

payment $K = PR_1 + I_1 = 156 + 43.1 = 199.1$ $OB_2 = OB_1(1+i) - K = 542.20$. $PR_2 = K - I_2 = 163.80$.

$I_3 = OB_2 \cdot i = 27.11$ $PR_3 = K - I_3 = 199.1 - 27.11 = 171.99$. $OB_3 = OB_2(1+i) - K = 370.21$.

$I_4 = OB_3 \cdot i = 18.51$. $PR_4 = K - I_4 = 180.59$. $OB_4 = OB_3(1+i) - K = 189.62$.

$I_5 = OB_4 \cdot i = 9.98$, $PR_5 = K - I_5 = 189.62$. $OB_5 = OB_4(1+i) - K = 0$.

3.2.4.

Assume monthly payment is 1.

$L = OB_0 = 1 \cdot a_{\overline{60}|1\%} = a_{\overline{60}|1\%}$, $OB_t = a_{\overline{60-t}|1\%}$. by Table 3.6.

$a_{\overline{60-t}|1\%} < \frac{a_{\overline{60}|1\%}}{2} \Rightarrow t = 34.4$, we take $t = 35$. (June 1, 2007).